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PROJECTIVE EMBEDDINGS OF THE TEICHMÜLLER SPACES OF BORDERED RIEMANN SURFACES

YOHEI KOMORI

ABSTRACT. We will show that except few cases, by using the hyperbolic length functions of simple closed geodesics, we can embed the Teichmüller space of a bordered Riemann surface into the real projective space of the same dimension. The key idea is to study the hyperbolic structure on a subsurface conformally isomorphic to a torus with a hole (named as a “cook-hat”), or a thrice-punctured sphere with a hole (named as a “crown”).

1. INTRODUCTION

Let M be a hyperbolic Riemann surface of genus g with n punctures and r holes. In this paper we assume that M has at least one boundary geodesic, i.e. $r \geq 1$. Then the Teichmüller space $\mathcal{T}_{g,n,r}$ is the space of isotopy classes of hyperbolic metrics on M which has a metric space structure homeomorphic to the real affine space $\mathbb{R}^{6g+2n+3r-6}$.

By using hyperbolic lengths of simple closed geodesics we can embed $\mathcal{T}_{g,n,r}$ into the real affine space. In practice we can embed $\mathcal{T}_{g,n,r}$ into $\mathbb{R}^{9g-9+3n+4r}$: Fix a pants decomposition \mathcal{P} on M , i.e. a multicurve such that $M \setminus \mathcal{P}$ is homeomorphic to the disjoint union of thrice punctured spheres. \mathcal{P} consists of $3g - 3 + n + r$ numbers of disjoint simple close curves. The Fenchel-Nielsen coordinates associate to each $m \in \mathcal{T}_{g,n,r}$ the length of each components of \mathcal{P} and boundary geodesics, and the twist of each components of \mathcal{P} , which is a diffeomorphism from $\mathcal{T}_{g,n,r}$ onto $\mathbb{R}_+^{3g-3+n+2r} \times \mathbb{R}^{3g-3+n+r}$ (see [IT]). On the other hand the twist of each components of \mathcal{P} can be determined by the lengths of two more curves for each components so that $\mathcal{T}_{g,n,r}$ can be embedded into $\mathbb{R}^{9g-9+3n+4r}$ by length functions of $9g - 9 + 3n + 4r$ number of simple closed geodesics. In his paper [S1], Schmutz showed that the minimal number of simple closed geodesics whose hyperbolic lengths globally parametrize $\mathcal{T}_{g,n,r}$ is equal to $\dim_{\mathbb{R}} \mathcal{T}_{g,n,r}$, so that the image of $\mathcal{T}_{g,n,r}$ in $\mathbb{R}^{\dim_{\mathbb{R}} \mathcal{T}_{g,n,r}}$ should be an unbounded domain.

Now we have the following natural question:

Can we find $\dim_{\mathbb{R}} \mathcal{T}_{g,n,r} + 1$ -number of simple closed geodesics whose hyperbolic lengths embed $\mathcal{T}_{g,n,r}$ into the finite dimensional real projective space $P(\mathbb{R}^{\dim_{\mathbb{R}} \mathcal{T}_{g,n,r} + 1})$?

Because of the PL-Structure of the Thurston boundary, we might expect that the image of $\mathcal{T}_{g,n,r}$ should be the interior of some convex polyhedron in $P(\mathbb{R}^{\dim_{\mathbb{R}} \mathcal{T}_{g,n,r} + 1})$.

In this paper we answer this question affirmatively except for the cases when $g = 0$ and $r = 0, 1, 2$. The key idea is to look for a subsurface homeomorphic to a thrice-punctured sphere with a hole or a torus with a hole, which is a tubular

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neighborhood of two geodesics contained in the members of geodesics parametrizing $T_{g,n,r}$ in $P(\mathbb{R}^{dim_{\mathbb{R}} T_{g,n,r+1}})$.

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2. REVIEW THE RESULTS OF SCHMUTZ

2.1. Surfaces with no handles. Let M be a Riemann surface of type $(0, n, r)$. From our assumption, n and r satisfy $n + r \geq 3$ and $r \geq 1$. We denote the boundary geodesics $x, a_1, a_2, \dots, a_{n+r-1}$ and dividing geodesics $b_1, b_2, \dots, b_{n+r-3}$ which decompose M into disjoint union of (degenerate) pair of pants (see Figure 1).

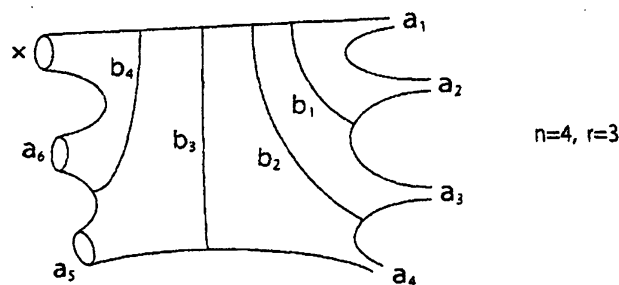


FIGURE 1

For each $i = 1, 2, \dots, n + r - 3$, let X_i be the subsurface of type $(0, n_i, r_i)$ where $n_i + r_i = 4$ with boundary geodesics $a_{i+1}, a_{i+2}, b_{i-1}, b_{i+1}$. Choose geodesics c_i and d_i in X_i so that the triple $\{b_i, c_i, d_i\}$ mutually intersect exactly twice. Then Schmutz proved that

Proposition 2.1. (cf. Proposition2 [S1])

The hyperbolic lengths of $2n + 3r - 6$ geodesics

$$a_1, a_2, \dots, a_{n+r-1}, b_1, c_1, c_2, c_{n+r-3}, d_1, d_2, d_{n+r-3}$$

embeds $T_{0,n,r}$ into $\mathbb{R}^{2n+3r-6}$. Here we remark that the length of a_k is equal to 0 when a_k corresponds to a puncture.

2.2. Surfaces with at least one handle. Next we consider a Riemann surface M of type (g, n, r) where $g \geq 1$.

First we consider the case $(g, 0, 1)$. We denote the boundary geodesic by x . Choose non-dividing geodesics $a_1, a_2, \dots, a_g, b_2, b_3, \dots, b_g, c_2, c_3, \dots, c_g$ which decompose M into disjoint union of pair of pants (see Figure 2).

For each $i = 2, \dots, g - 1$, let X_i be the subsurface of type $(0, 0, 4)$ with boundary geodesics $b_i, c_i, b_{i+1}, c_{i+1}$. Choose geodesics d_{i+1} and e_{i+1} in X_i so that the triple $\{a_{i+1}, d_{i+1}, e_{i+1}\}$ mutually intersect exactly twice. Let X_1 be the subsurface of M of type $(0, 0, 4)$ with boundary geodesics a_1, a_1, b_2, c_2 , and choose d_2 and e_2 on X_1 so that the triple $\{a_2, d_2, e_2\}$ mutually intersect exactly twice. Moreover let f be a geodesic intersecting with $a_1, b_2, b_3, \dots, b_g, c_2, c_3, \dots, c_g$ exactly once. Then for $i = 2, \dots, g$, we can find geodesics $r_1, s_2, s_3, \dots, s_g, t_2, t_3, \dots, t_g$ so that $\{a_1, r_1, f\}$, $\{b_i, s_i, f\}$ and $\{c_i, t_i, f\}$ mutually intersect exactly once. In this case, Schmutz proved that

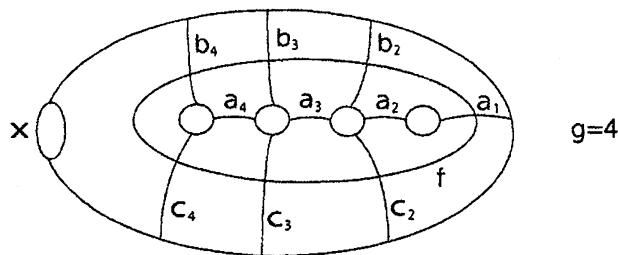


FIGURE 2

Proposition 2.2. (cf. Proposition 3 [S1])

The hyperbolic lengths of $6g - 3$ geodesics

$$a_1, a_2, \dots, a_g, b_1, \dots, b_g, d_1, \dots, d_g, e_1, \dots, e_g, f, r_1, s_1, \dots, s_g, t_1, \dots, t_g$$

embeds $T_{g,0,1}$ into \mathbb{R}^{6g-3} .

Finally we consider a Riemann surface M of type (g, n, r) where $g \geq 1$ in general. First we choose a dividing geodesic x to decompose M into subsurfaces M' of type $(g, 0, 1)$ and N' of type $(0, n, r+1)$ (see Figure 3).

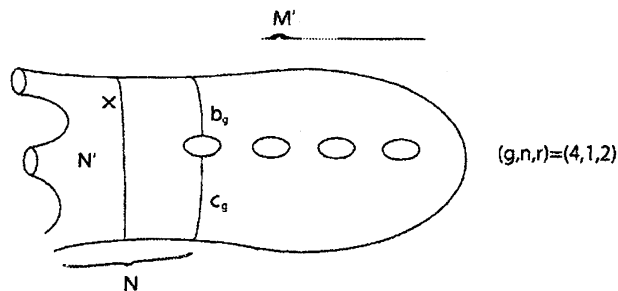


FIGURE 3

Let N be the subsurface of M consisting of N' and the pair of pants whose boundary curves are x, b_g and c_g . Then from the above argument we can choose $6g - 3$ curves from M' and $2n + 3(r+2) - 6$ curves from N which determines M' and N in $T_{g,0,1}$ and $T_{0,n,r+2}$ respectively. On the other hand the lengths of curves x, b_g and c_g are counted twice in M' and N so that we can find $6g - 3 + 2n + 3(r+2) - 6 - 3 = 6g + 2n + 3r - 6$ geodesics whose hyperbolic lengths embed $T_{g,n,r}$ into $\mathbb{R}^{6g+2n+3r-6}$.

3. MAIN RESULT

First let M be a Riemann surface of type $(0, n, r)$. We assume that $n \geq 3$ and a_1, a_2, a_3 are punctures. Then the subsurface X_1 bounded by a_1, a_2, a_3 and b_1 is a thrice-punctured sphere with a hole, on which the triple $\{b_1, c_1, d_1\}$ mutually intersect exactly twice (see Figure 1). Therefore by means of Corollary 5.6, the hyperbolic lengths of $2n + 3r - 5$ geodesics

$$a_1, a_2, \dots, a_{n+r-1}, b_1, c_1, c_2, c_{n+r-3}, d_1, d_2, d_{n+r-3}, b_2$$

embeds $T_{0,n,r}$ into $P(\mathbb{R}^{2n+3r-5})$.

Next we suppose M is a Riemann surface of type (g, n, r) where $g \geq 1$. Then there is a subsurface X of M with a geodesic boundary, which is a tubular neighborhood of the union of geodesics a_1 and f . X is homeomorphic to a torus with a hole on which the triple $\{a_1, r_1, f\}$ mutually intersect exactly once (see Figure 2). Then by means of Theorem 4.4, the proportion of the hyperbolic lengths of $6g + 2n + 3r - 5$ geodesics embeds $T_{g,n,r}$ into $P(\mathbb{R}^{6g+2n+3r-5})$.

Summarizing the above arguments,

Theorem 3.1. *Assume that $g \geq 1$ or $n \geq 3$. Then the Teichmüller space $T_{g,n,r}$ of a bordered Riemann surface can be embedded into the real projective space of $\dim_{\mathbb{R}} T_{g,n,r}$ by the hyperbolic length functions of $\dim_{\mathbb{R}} T_{g,n,r} + 1$ simple closed geodesics.*

For a sphere (i.e., $g = 0$) with holes (i.e., $r \geq 1$), this question is still open for the cases $n = 0, 1, 2$.

4. COOK-HATS

In this section we will consider complete hyperbolic structures on a torus with a hole. We call a hyperbolic torus with a hole a **cook-hat**.

Definition 4.1. Three simple closed geodesics (α, β, γ) on a cook-hat is called a **canonical triple** if each pair of them has the intersection number equal to one.

We remark that the hyperbolic lengths of a canonical triple (α, β, γ) satisfy triangle inequalities.

For the hyperbolic lengths of a canonical triple (α, β, γ) and the boundary geodesic δ on a cook-hat, we have the following equality and inequality.

Proposition 4.2. *For any cook-hat with the boundary geodesic δ and a canonical triple (α, β, γ) , their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following equality and inequality:*

$$(4.1) \quad \cosh^2 \frac{l(\delta)}{4} = (\cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta) - l(\gamma)}{2}).$$

$$(4.2) \quad l(\alpha) + l(\beta) + l(\gamma) > l(\delta).$$

Proof. We uniformize a cook-hat by a Fuchsian group $\Gamma \subset SL(2, \mathbb{R})$, and denote the traces of elements representing α, β, γ and δ by $t(\alpha), t(\beta), t(\gamma)$ and $t(\delta)$. Then they satisfy

$$(4.3) \quad t(\delta) - 2 = t(\alpha)t(\beta)t(\gamma) - (t(\alpha)^2 + t(\beta)^2 + t(\gamma)^2).$$

By means of the relation between trace functions and length functions

$$(4.4) \quad |t(\alpha)| = 2 \cosh \frac{l(\alpha)}{2}$$

and the equality

$$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y),$$

we can rewrite (4.3) in terms of length functions

$$\begin{aligned}
2 \cosh \frac{l(\delta)}{2} - 2 &= t(\delta) - 2 \\
&= t(\alpha)t(\beta)t(\gamma) - (t(\alpha)^2 + t(\beta)^2 + t(\gamma)^2) \\
&= 4(2 \cosh \frac{l(\alpha)}{2} \cosh \frac{l(\beta)}{2} \cosh \frac{l(\gamma)}{2} - \cosh^2 \frac{l(\alpha)}{2} - \cosh^2 \frac{l(\beta)}{2} - \cosh^2 \frac{l(\gamma)}{2}) \\
&= 4(\cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta) - l(\gamma)}{2}) - 4.
\end{aligned}$$

Therefore

$$\begin{aligned}
\cosh^2 \frac{l(\delta)}{4} &= \frac{1}{2}(\cosh \frac{l(\delta)}{2} + 1) \\
&= (\cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta) - l(\gamma)}{2})
\end{aligned}$$

which is the equality (4.1).

Since $\cosh x$, hence $\cosh^2 x$ is monotonely increasing function of x , the equality (4.1) implies that it is enough to show that

$$(\cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta) - l(\gamma)}{2}) < \cosh^2 \frac{l(\alpha) + l(\beta) + l(\gamma)}{4}$$

for the proof of the inequality (4.2). In practice

$$\begin{aligned}
&\cosh^2 \frac{l(\alpha) + l(\beta) + l(\gamma)}{4} \\
&\quad - (\cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2})(\cosh \frac{l(\alpha)}{2} - \cosh \frac{l(\beta) - l(\gamma)}{2}) \\
&= \cosh^2 \frac{l(\alpha) + l(\beta) + l(\gamma)}{4} + \cosh^2 \frac{l(\alpha)}{2} + \cosh \frac{l(\beta) + l(\gamma)}{2} \cosh \frac{l(\beta) - l(\gamma)}{2} \\
&\quad - \cosh \frac{l(\alpha)}{2} \cosh \frac{l(\beta) + l(\gamma)}{2} - \cosh \frac{l(\alpha)}{2} \cosh \frac{l(\beta) - l(\gamma)}{2} \\
&= \frac{1}{4} \{ (e^{l(\alpha)} - e^{\frac{l(\alpha) + l(\beta) - l(\gamma)}{2}}) + (e^{l(\beta)} - e^{\frac{l(\beta) + l(\gamma) - l(\alpha)}{2}}) + (e^{l(\gamma)} - e^{\frac{l(\gamma) + l(\alpha) - l(\beta)}{2}}) \\
&\quad + (1 - e^{\frac{l(\alpha) - l(\beta) - l(\gamma)}{2}}) + (1 - e^{\frac{l(\beta) - l(\gamma) - l(\alpha)}{2}}) + (1 - e^{\frac{l(\gamma) - l(\alpha) - l(\beta)}{2}}) \\
&\quad + e^{-l(\alpha)} + e^{-l(\beta)} + e^{-l(\gamma)} + 1 \} > 0.
\end{aligned}$$

□

Remark 4.3. (1) The equality (4.1) also follows from the plane hyperbolic geometry of the right angled hexagon which is the symmetric half of the pair of pants $T \setminus \alpha$.

(2) The inequality (4.2) also comes from the fact that the curve $\alpha \cup \beta \cup \gamma$ is freely homotopic to the geodesic δ .

By means of the equality (4.1) in Proposition 4.2, we can embed the Teichmüller space $\mathcal{T}(T)$ of a torus with a hole into the 3-dimensional real projective space $P(\mathbb{R}^4)$.

Theorem 4.4. *For a cook hat with a canonical triple (α, β, γ) and the boundary geodesic δ , their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy*

$$\cosh^2 \frac{sl(\delta)}{4} < (\cosh \frac{sl(\beta) + sl(\gamma)}{2} - \cosh \frac{sl(\alpha)}{2})(\cosh \frac{sl(\alpha)}{2} - \cosh \frac{sl(\beta) - sl(\gamma)}{2})$$

for any $s > 1$. In particular the system of length functions $L := (l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $T(T)$ of a torus with a hole into $P(\mathbb{R}^4)$.

Proof. For simplicity we will write

$$a = l(\alpha), b = l(\beta), c = l(\gamma), d = l(\delta).$$

Then our claim is rewritten as

$$\frac{d}{4}s < \cosh^{-1} \sqrt{f(s)}, \quad \forall s > 1$$

where

$$f(s) := (\cosh \frac{b+c}{2}s - \cosh \frac{a}{2}s)(\cosh \frac{a}{2}s - \cosh \frac{b-c}{2}s),$$

for which it is enough to show that

$$\frac{d}{ds} \cosh^{-1} \sqrt{f(s)} > \frac{d}{4}, \quad \forall s > 1.$$

By the inequality (4.2), it is enough to show that

$$\frac{d}{ds} \cosh^{-1} \sqrt{f(s)} > \frac{a+b+c}{4}, \quad \forall s > 1.$$

By the following simple estimation

$$\frac{d}{ds} \cosh^{-1} \sqrt{f(s)} = \frac{f'(s)}{2\sqrt{f(s)}\sqrt{f(s)-1}} > \frac{f'(s)}{2f(s)}$$

we will show that

$$\frac{f'(s)}{f(s)} > \frac{a+b+c}{2}, \quad \forall s > 1.$$

In practice

$$\begin{aligned} \frac{f'(s)}{f(s)} &= \frac{\frac{d}{ds}(\cosh \frac{b+c}{2}s - \cosh \frac{a}{2}s)}{\cosh \frac{b+c}{2}s - \cosh \frac{a}{2}s} + \frac{\frac{d}{ds}(\cosh \frac{a}{2}s - \cosh \frac{b-c}{2}s)}{\cosh \frac{a}{2}s - \cosh \frac{b-c}{2}s} \\ &> \frac{b+c}{2} + \frac{a}{2} = \frac{a+b+c}{2}. \end{aligned}$$

Here we use the following lemma:

Lemma 4.5. For $0 < p < q$,

$$g(s) := \frac{\frac{d}{ds}(\cosh qs - \cosh ps)}{\cosh qs - \cosh ps} = \frac{q \sinh qs - p \sinh ps}{\cosh qs - \cosh ps} > q, \quad \forall s > 1.$$

Proof. It is enough to show that the derivative of $g(s)$ is negative for $\forall s > 1$, since

$$\lim_{s \rightarrow \infty} g(s) = \lim_{s \rightarrow \infty} \frac{q \sinh qs - p \sinh ps}{\cosh qs - \cosh ps} = q.$$

Hence we will show the negativity of the numerator of $g'(s)$:

$$g'(s) = \frac{(q^2 \cosh qs - p^2 \cosh ps)(\cosh qs - \cosh ps) - (q \sinh qs - p \sinh ps)^2}{(\cosh qs - \cosh ps)^2}.$$

In practice

$$\begin{aligned}
& (q^2 \cosh qs - p^2 \cosh ps)(\cosh qs - \cosh ps) - (q \sinh qs - p \sinh ps)^2 \\
&= q^2 \cosh^2 qs + p^2 \cosh^2 ps - (q^2 + p^2) \cosh qs \cosh ps \\
&\quad - q^2 \sinh^2 qs - p^2 \sinh^2 ps + 2pq \sinh qs \sinh ps \\
&= q^2 + p^2 - \frac{1}{2}(q+p)^2 \cosh(q-p)s - \frac{1}{2}(q-p)^2 \cosh(q+p)s \\
&< q^2 + p^2 - \frac{1}{2}(q+p)^2 - \frac{1}{2}(q-p)^2 = 0.
\end{aligned}$$

□

□

By means of the triangle inequalities of $l(\alpha), l(\beta), l(\gamma)$ and the inequality (4.2) in Proposition 4.2, we can determine the image of $\mathcal{T}(T)$ in $\mathcal{P}(\mathbb{R}^4)$ as follows.

Theorem 4.6. *The image of $\mathcal{T}(T)$ the Teichmüller space of a cook-hat under the map $L := (l(\alpha) : l(\beta) : l(\gamma) : l(\delta))$ is the convex polyhedron Δ in $\mathcal{P}(\mathbb{R}^4)$ defined by*

$$\begin{aligned}
\Delta := \{ & (a : b : c : d) \in \mathcal{P}(\mathbb{R}^4) \mid a > 0, b > 0, c > 0, d > 0, \\
& a < b + c, b < c + a, c < a + b, d < a + b + c \}.
\end{aligned}$$

Proof. By means of the inequality (4.2) in Proposition 4.2, we have $L(T) \subset \Delta$. Hence we will prove that $\Delta \subset L(T)$. Take any point $p \in \Delta$ and four positive real numbers $(a, b, c, d) \in \mathbb{R}_+^4$ satisfying $p = (a : b : c : d)$. Then there exist $s > 0$ and a hyperbolic structure $m \in \mathcal{T}(T)$ such that

$$(l(\alpha), l(\beta), l(\gamma), l(\delta)) = (as, bs, cs, d_s)$$

where $l(\alpha) = l(m, \alpha)$ and $d_s > 0$ is defined by

$$d_s := 4 \cosh^{-1} \sqrt{(\cosh \frac{sb+sc}{2} - \cosh \frac{sa}{2})(\cosh \frac{sa}{2} - \cosh \frac{sb-sc}{2})}.$$

To conclude that $L(m) = p$, It is enough to show that there is $s > 0$ such that $d_s = sd$. We will show that d_s/s takes any value between 0 and $a+b+c$ when s varies. In practice d_s/s is a continuous function on s and

$$(\cosh \frac{sb+sc}{2} - \cosh \frac{sa}{2})(\cosh \frac{sa}{2} - \cosh \frac{sb-sc}{2}) \rightarrow 1$$

when s decreases, hence $d_s/s \rightarrow 0$. On the other hand,

$$\begin{aligned}
& (\cosh \frac{sb+sc}{2} - \cosh \frac{sa}{2})(\cosh \frac{sa}{2} - \cosh \frac{sb-sc}{2}) \\
&= e^{\frac{(a+b+c)s}{2}} O(1), \quad s \rightarrow \infty
\end{aligned}$$

and

$$\cosh \frac{d_s}{4} = e^{\frac{d_s}{4}} O(1), \quad s \rightarrow \infty$$

imply that $\lim_{s \rightarrow \infty} d_s/s = a + b + c$. Hence d_s/s takes any value between 0 and $a + b + c$. □

5. CROWNS

In this section we will consider complete hyperbolic structures on a thrice-punctured sphere with a hole. We call a hyperbolic thrice-punctured sphere with a hole a **crown**.

Definition 5.1. Three simple closed geodesics (α, β, γ) on a crown is called a **canonical triple** if each pair of them has the intersection number equal to two.

We will show that similar results in section 2 also hold for $\mathcal{T}(S)$ the Teichmüller space of a thrice-punctured sphere with a hole with the help of the geometric bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$ explained below. For this purpose we realize $\mathcal{T}(T)$ and $\mathcal{T}(S)$ as hypersurfaces in \mathbb{R}^4 in terms of trace functions:

Theorem 5.2. (Theorem 2 of [L] and Proposition 3.1 of [NN])

- (1) *We uniformize a cook-hat $m \in \mathcal{T}(T)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_T : \mathcal{T}(T) \rightarrow \mathbb{R}^4$ defined by $\varphi_T(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image $\varphi_T(\mathcal{T}(T))$ is described as follows:*

$$\{(a, b, c, d) \in \mathbb{R}^4 \mid a > 2, b > 2, c > 2, d > 2, \\ abc - a^2 - b^2 - c^2 + 2 = d\}.$$

- (2) *We uniformize a crown $m \in \mathcal{T}(S)$ by a Fuchsian group and denote the traces of elements representing a canonical triple α, β, γ and boundary geodesic δ by $t_\alpha(m), t_\beta(m), t_\gamma(m)$ and $t_\delta(m)$. Then the map $\varphi_S : \mathcal{T}(S) \rightarrow \mathbb{R}^4$ defined by $\varphi_S(m) := (t_\alpha(m), t_\beta(m), t_\gamma(m), t_\delta(m))$ is injective and the image $\varphi_S(\mathcal{T}(S))$ is described as follows:*

$$\{(p, q, r, s) \in \mathbb{R}^4 \mid p > 2, q > 2, r > 2, s > 2, s^2 + 2(p + q + r + 4)s \\ + 4(p + q + r) + p^2 + q^2 + r^2 - pqr + 8 = 0\}.$$

Then by means of trace functions, we have the following geometric bijection between $\mathcal{T}(T)$ and $\mathcal{T}(S)$:

Theorem 5.3. *There is a bijection from $\mathcal{T}(T)$ to $\mathcal{T}(S)$ which sends a cook-hat T with the lengths of a canonical triple and the boundary geodesic equal to (l_1, l_2, l_3, l_4) to a crown S with the lengths of a canonical triple and the boundary geodesic equal to $(2l_1, 2l_2, 2l_3, l_4)$.*

Proof. When we substitute $(a^2 - 2, b^2 - 2, c^2 - 2, d)$ for (p, q, r, s) , the equation $s^2 + 2(p + q + r + 4)s + 4(p + q + r) + p^2 + q^2 + r^2 - pqr + 8$ factorizes as

$$d^2 + 2(p + q + r + 4)d + 4(p + q + r) + p^2 + q^2 + r^2 - pqr + 8 \\ = (d - (abc - a^2 - b^2 - c^2 + 2))(d - (-abc - a^2 - b^2 - c^2 + 2)).$$

Hence the map $\Psi : \varphi_T(\mathcal{T}(T)) \rightarrow \varphi_S(\mathcal{T}(S))$ defined by $\Psi(a, b, c, d) := (a^2 - 2, b^2 - 2, c^2 - 2, d)$ is bijective. Also the relation between trace functions and length functions

$$|t(\alpha)| = 2 \cosh \frac{l(\alpha)}{2}$$

tells us the length relations between $m \in \mathcal{T}(T)$ and $\varphi_S^{-1} \circ \Psi \circ \varphi_T(m) \in \mathcal{T}(S)$. \square

Remark 5.4. For the limiting case $l(\delta) = 0$, this bijection reduces to the well-known correspondence between punctured tori and forth-punctured spheres, which follows from the commensurability of uniformizing Fuchsian groups (see [ASWY]).

This bijection induces the next corollaries: The following inequality is the counterpart of the inequality (4.2) in Proposition 4.2 for crowns.

Corollary 5.5. *For any crown with the boundary geodesic δ and a canonical triple (α, β, γ) , their hyperbolic lengths $l(\alpha), l(\beta), l(\gamma)$ and $l(\delta)$ satisfy the following inequality:*

$$l(\alpha) + l(\beta) + l(\gamma) > 2l(\delta).$$

Next result is the counterpart of Theorem 4.4 and 4.6 for crowns.

Corollary 5.6. *For a crown with a canonical triple (α, β, γ) and the boundary geodesic δ , the system of length functions $(l(\alpha), l(\beta), l(\gamma), l(\delta))$ gives a homogeneous coordinate of the Teichmüller space $T(S)$ into $P(\mathbb{R}^4)$. The image of $T(S)$ is the convex polyhedron in $P(\mathbb{R}^4)$ defined by*

$$\{(a : b : c : d) \in P(\mathbb{R}^4) \mid a > 0, b > 0, c > 0, d > 0, \\ a < b + c, b < c + a, c < a + b, 2d < a + b + c\}.$$

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